ON THE PROBLEM OF STABILITY OF THE SOLUTION OF THE NONLINEAR HEAT-CONDUCTION EQUATION

(K VOPROSU OB USTOICHIVOSTI RESHENIIA NELINEINOGO URAVNENIIA TEPLOPROVODNOSTI)

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A series of papers has recently been devoted to the investigation of the stability of the solution of the nonlinear heat-conduction equation under a number of circumstances. In particular, in [1] the author investigated the stability of the solution of such an equation in the metric of space L_2. In what follows we shall prove that the solution of a somewhat more general problem is stable in the same metric. In this, by the use of the methods of [2], it is possible to show that there exists a δ, such that if for the solution $u(t, x)$ of the equation under consideration we have $\| u(0, x) \| < \delta$, then $\| u(t, x) \| \to 0$ as $t \to \infty$.

Let G be a bounded domain of space $R^m(x_1, \ldots, x_m)$ and let Γ be its boundary. We consider the problem

$$\frac{\partial u}{\partial t} = Lu + f(t, x, u), \qquad (t, x) \in [0, \infty) \times G u|_\Gamma = 0, \ u(0, x) = \varphi(x) \tag{1}$$

Here

$$Lu \equiv \sum_{i,k=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ik}(x)\frac{\partial u}{\partial x_k}\right) + \sum_{i=1}^{m} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

represents such an elliptic operator in G, that for some $\lambda > 0$ we have

$$\int_G u \cdot Lu\, dx \leqslant -\lambda \int_G u^2\, dx, \qquad u|_\Gamma = 0 \tag{2}$$

We introduce in the group $M = \{u(t, x)\}$ of continuous functions, possessing continuous derivatives

$$\frac{\partial u}{\partial t}, \quad \frac{\partial^2 u}{\partial x_k \partial x_j} \qquad (k, j = 1, \ldots, m)$$

the scalar product

$$(u, v) = \int_G uv\,dx, \qquad \|u\| = \sqrt{(u, u)}$$

Let $u(t, x)$ be the solution of the problem

$$\frac{\partial u}{\partial t} = Lu, \qquad u|_\Gamma = 0, \qquad u(0, x) = \varphi(x)$$

We denote by $T(t)$ an operator which is defined by the equation $u(t, x) = T(t)\phi(x)$. If certain requirements concerning the smoothness of coefficients and the boundary of the domain are satisfied, which will be assumed, then problem (1) can be written in the form of the integral (see, for example [3, 4])

$$u(t, x) = T(t)\varphi(x) + \int_0^t T(t-s) f(s, x, u(s, x))\,ds \tag{3}$$

Later, we shall require an estimate of the norm of operator $T(t)$. Multiplying the equation

$$\frac{\partial T(t)\varphi}{\partial t} = LT(t)\varphi$$

by $T(t)$ scalarly, we shall obtain

$$\frac{1}{2}\frac{d}{dt}\|T(t)\varphi\|^2 \leqslant -\lambda\|T(t)\varphi\|^2, \qquad \|T(0)\varphi\| = \|\varphi\|.$$

in view of (2). Hence, applying the theorem on the differential inequality [5],

$$\|T(t)\varphi\|^2 \leqslant \|\varphi\|^2 e^{-2\lambda t}$$

we have

$$\|T(t)\| \leqslant e^{-\lambda t} \tag{4}$$

(5)

Theorem. For $\|u\| \leqslant \gamma$ and $t \geqslant 0$ let

$$\|f(t, x, u)\| \leqslant k\|u\| + \psi(t)\|u\|^{1+\alpha} \quad \alpha > 0, \qquad <k> \lambda, \qquad \int_0^\infty e^{\alpha(k-\lambda)t}\psi(t)\,dt < \infty$$

Then, irrespective of the value of the positive number $\epsilon \leqslant \gamma$, with

$$\|\varphi\| < \delta = \left[\varepsilon^{-\alpha} + \alpha\int_0^\infty e^{\alpha(k-\lambda)t}\psi(t)\,dt\right]^{-1/\alpha} \tag{6}$$

the solution $u(t, x)$ of Equation (1) satisfies the inequality

$$\|u(t, x)\| < \varepsilon e^{(k-\lambda)t} \qquad (t \geqslant 0) \tag{7}$$

which means that the trivial solution of (1) (for $\phi = 0$) is asymptotically stable.

Proof. We shall first show that for any $t \geqslant 0$ the inequality

$$\| u(t, x) \| < \gamma \tag{8}$$

is satisfied. Let t_0 be the first point of inequality (8). With $t \in [0, t_0]$, we have

$$\| u(t, x) \| < \delta e^{-\lambda t} + \int_0^t e^{-\lambda (t-s)} \{ k \| u(s, x) \| + \psi(s) \| u(s, x) \|^{1+\alpha} \} ds \tag{9}$$

It follows from (9) by the application of the lemma on the integral inequality [2] that $\| u(t, x) \| < v(t)$ for $t \in [0, t_0]$. Here $v(t)$ denotes the solution of the equation

$$v(t) = \delta e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} \{ kv(s) + \psi(s) v^{1+\alpha}(s) \} ds$$

Introducing a new unknown $z(t) = v(t) e^{\lambda t}$ and differentiating both sides of this equation, we shall obtain

$$\frac{dz}{dt} = kz + e^{-\alpha \lambda t} \psi(t) z^{1+\alpha}, \qquad z(0) = \delta$$

from Bernoulli's equation. Solving this equation we shall find that

$$v(t) = e^{(k-\lambda)t} \left\{ \delta^{-\alpha} - \alpha \int_0^t e^{\alpha(k-\lambda)s} \psi(s) ds \right\}^{-1/\alpha}$$

In accordance with the definition of δ, we have

$$\delta^{-\alpha} - \alpha \int_0^t e^{\alpha(k-\lambda)s} \psi(s) ds \geqslant \varepsilon^{-\alpha}$$

Hence

$$\| u(t, x) \| < v(t) \leqslant \varepsilon e^{(k-\lambda)t} \text{ for } t \in [0, t_0] \tag{10}$$

and, consequently $\| u(t_0, x) \| < \epsilon e^{(k-\lambda) t_0} < \gamma$, which contradicts the definition of point t_0. It follows that inequality (8), and so also (5), is satisfied at any $t \geqslant 0$, and, hence, also inequality (10) which is a consequence of the former. This proves the theorem.

Note. Making use of the same method, it is possible to prove the theorem for the case when the coefficients of Equation (1), λ in inequality (2), and k in condition (5) depend on t. For this purpose it is sufficient to substitute

$$\int_0^t [k(s) - \lambda(s)]\, ds$$

for

$$(k - \lambda)\, t$$

in the statement of the theorem.

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